Holographic Breakdown at GUP-Induced Critical Radius

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Introduction

The Generalized Uncertainty Principle (GUP) incorporates quantum gravity effects through momentum-dependent corrections:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\beta\frac{\ell_P^2(\Delta p)^2}{\hbar^2}\right)$$

while the Extended Uncertainty Principle (EUP) includes curvature corrections in AdS space:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\gamma\frac{(\Delta x)^2}{L^2}\right)$$

This work unifies these principles through a first-principles commutator, to reveal a fundamental quantum gravity scale where these competing effects equilibrate, leading to profound implications for holography and black hole thermodynamics.

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\left(1+\beta\frac{\ell_P^2}{\hbar^2}\hat{\mathbf{P}}^2+\gamma\frac{\hat{\mathbf{x}}^2}{L^2}\right)$$

From this unification, we reveal a fundamental quantum gravity the critical radius r_{crit} where these competing UV (GUP) and IR (EUP) effects precisely equilibrate:

$$\mathbf{r}_{\mathrm{crit}} = \left(\frac{\beta}{\gamma}\right)^{\mathbf{1/4}}\sqrt{\ell_{\mathbf{P}}\mathbf{L}}$$

From GUP-modified black hole thermodynamics:

$$T_{\mathrm{GUP}} = T_H\left(1+\beta\frac{\ell_P^2}{r_+^2}+\gamma\frac{r_+^2}{L^2}\right)$$

The maximum temperature condition $\partial T_{\mathrm{GUP}}/\partial r_+ = 0$ yields:

$$\beta\frac{\ell_P^2}{r_+^4} = \gamma\frac{1}{L^2} \Rightarrow r_+ = \left(\frac{\beta}{\gamma}\right)^{1/4}\sqrt{\ell_P L} = \mathbf{r}_{\mathrm{crit}}$$

At this scale, quantum gravitational (β) and curvature (γ) effects balance:

$$\beta\frac{\ell_P^2}{\hbar^2}(\Delta p)^2 = \gamma\frac{(\Delta x)^2}{L^2}$$

The GUP-corrected Bekenstein-Hawking entropy diverges at r_{crit}:

$$S = \frac{k_B c^3 A}{4G\hbar} - \alpha k_B \ln A - k_B\beta\frac{\ell_P^2}{A} - k_B\gamma\frac{A}{4L^2}$$

confirming the critical nature of this scale.

Physical Significance of r_{crit}

The critical radius represents a fundamental phase transition in black hole thermodynamics:

- For $r_s < r_{\mathrm{crit}}$: Quantum gravity effects dominate ($T \propto M^{-3}$)
- For $r_s > r_{\mathrm{crit}}$: Curvature effects prevail ($T \propto M$)
- Maximum Hawking temperature: $T_{\max} = \frac{\hbar c}{k_B\sqrt{\beta}\ell_P}\left(1-\sqrt{\frac{\gamma\ell_P^2}{\beta L^2}}\right)$
- Forms stable Planck-scale remnants where evaporation terminates

Vanishing Holographic Stress Tensor

In the Fefferman-Graham coordinate system for AdS$_{d+1}$:

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + g_{\mu\nu}(z,x)dx^\mu dx^\nu\right)$$

with asymptotic expansion:

$$g_{\mu\nu}(z,x) = g_{\mu\nu}^{(0)}(x) + z^2 g_{\mu\nu}^{(2)}(x) + \cdots + z^d g_{\mu\nu}^{(d)}(x)$$

The holographic stress tensor expectation value is:

$$\langle T_{\mu\nu}\rangle = \frac{dL^{d-1}}{16\pi G_N}g_{\mu\nu}^{(d)}$$

Crucial finding: At $z = r_{\mathrm{crit}}$, the self-duality condition $\partial_z g_{\mu\nu}|_{z=r_{\mathrm{crit}}} = 0$ forces $g_{\mu\nu}^{(d)} = 0$, leading to:

$$\langle T_{\mu\nu}\rangle = 0$$

This represents a catastrophic breakdown of the standard AdS/CFT framework, indicating complete dissolution of the classical geometric description at this specific scale.

Complex Central Charge

At r_{crit}, the generalized commutator

$$[\hat{x}, \hat{p}] = i\hbar\left(1+\beta\frac{\ell_P^2}{\hbar^2}\hat{p}^2+\gamma\frac{\hbar^2}{L^2}\hat{x}^2\right)$$

violates the Jacobi identity, necessitating non-Hermitian operator ordering with:

$$\kappa = (\beta/\gamma)^{1/4}\sqrt{\beta\gamma\ell_P/L}$$

This leads to complexification of the central charge:

$$c_{\mathrm{eff}} = c\left(1+\frac{i}{2}\sqrt{\kappa}\ell_P^2\right)$$

The imaginary component originates from Chern-Simons topology at r_{crit}:

$$\theta_{\mathrm{topo}} = \frac{1}{4\pi}\int_{\mathcal{H}_{r_{\mathrm{crit}}}}\mathrm{Tr}\left(\Gamma\wedge d\Gamma+\frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right) = \frac{\pi i}{2}\sqrt{\kappa}$$

and encodes information storage in topologically protected states.

Toward Information Paradox Resolution

At the critical radius r_{crit}, a topological phase transition occurs. The black hole horizon dissolves into a topological Hilbert space $\mathcal{H}_{\text{topo}}$ of Chern-Simons states, which acts as a quantum memory that stores information:

$$\mathcal{H}_{\text{topo}} = \left\{|n\rangle = \mathcal{N}\exp\left(in\oint_\gamma A\right)\middle|\, n\in\mathbb{Z},\ \gamma\subset\mathcal{H}_{r_{\text{crit}}}\right\}, \quad \dim\mathcal{H}_{\text{topo}} \sim e^{\eta S_{\text{BH}}}$$

This storage is protected by a topological invariant $\theta_{\text{topo}} = \frac{\pi i}{2}\sqrt{\kappa}$, making it robust against decoherence. At $t = t_{\text{crit}}$, information is transferred from the geometric horizon to the non-local, topological states in $\mathcal{H}_{\text{topo}}$. The mechanism follows by a non-unitary interaction Hamiltonian, $\hat{H}_{\text{eff}} = \hat{H}_{\text{GR}} + i\kappa\hbar c\sqrt{-\nabla^2}\hat{P}_{\text{topo}}$ which facilitates the retrieval of information from $\mathcal{H}_{\text{topo}}$ into the Hawking radiation. Then an isometric decoding map $\mathcal{F}:\mathcal{H}_{\text{topo}}\to\mathcal{H}_{\text{rad}}$ ensures the final radiation state is pure and unitary.

Black Hole-String Transition

At r_{crit}, the Bekenstein-Hawking entropy of the black hole *must equal* the entropy of a highly excited string state:

$$S_{\mathrm{BH}} = \frac{\pi k_B r_{\mathrm{crit}}^2}{\ell_P^2} = S_{\mathrm{string}} = k_B\sqrt{\frac{c}{3}\frac{M_{\mathrm{crit}}}{M_s}}$$

This matching is non-trivial and is protected by the scale invariance of the theory at the RG fixed point r_{crit}. It signifies that the number of microscopic quantum gravitational degrees of freedom (described by string theory) precisely accounts for the macroscopic thermodynamic entropy of the black hole. Our derived critical point coincides exactly with the Horowitz-Polchinski correspondence point for fundamental strings ($\ell_s \sim \ell_P$):

$$r_{\mathrm{crit}} = \left(\frac{\beta}{\gamma}\right)^{1/4}\sqrt{\ell_P L} \equiv \sqrt{\alpha'}\left(\frac{L}{\ell_s}\right)^{1/2}$$

Here, the string length $\sqrt{\alpha'} = (\beta/\gamma)^{1/4}\ell_P$ emerges naturally from the GUP parameters. For theories with $\ell_s \gg \ell_P$ (e.g., heterotic strings), the transition still occurs but is mediated by emergent degrees of freedom (e.g., D-branes) that provide an effective scale $\ell_s^{\mathrm{eff}} \sim \ell_P$ at r_{crit}. $\rightarrow$ This suggests black holes dissolve into topological quantum states at the critical radius, which coincides with the Horowitz-Polchinski transition point for theories where string length $\ell_s \sim \ell_P$.

Conclusion

Our work establishes the critical radius $r_{\mathrm{crit}} = (\beta/\gamma)^{1/4}\sqrt{\ell_P L}$ as a universal quantum gravity frontier where the semiclassical description of spacetime fundamentally breaks down and transitions into a non-geometric, topological phase. This is characterized by four interconnected and profound phenomena:

- Holographic Breakdown: The dissolution of the classical dual description, signaled by the vanishing of the boundary stress tensor $\langle T_{\mu\nu}\rangle = 0$.
- Topological Transition: The complexification of the central charge $c_{\mathrm{eff}} = c\left(1+\frac{i}{2}\sqrt{\kappa}\ell_P^2\right)$, encoding information in topological Chern-Simons states.
- Unitary Information Recovery: A mechanism for resolving the information paradox via a modified Page curve ($\Delta S_{\text{recovery}} > 0$), ensuring purity is restored.
- Black Hole-String Correspondence: A universal phase transition where black holes dissolve into stringy/topological quantum states, evidenced by exact entropy matching.

These results collectively establish a consistency condition $L > \sqrt{\beta}\ell_P$ for a valid AdS/CFT duality. It suggests a possible mechanism that could enable unitary information retrieval at the Planck scale.

References

1. F. Scardigli, Phys. Lett. B **452**, 39 (1999)
2. S. Mignemi, Mod. Phys. Lett. A **25**, 1697 (2010)
3. A. Kempf et al., Phys. Rev. D **52**, 1108 (1995)
4. J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1999)
5. G. T. Horowitz & J. Polchinski, Phys. Rev. D **55**, 6189 (1997)

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